V. SUBSPACES AND ORTHOGONAL PROJECTION

In this chapter we will discuss the concept of subspace of Hilbert space, introduce a series of subspaces related to Haar wavelet, explore the orthogonal projection onto Hilbert subspaces.

Definition 1. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}$ be a linear subspace of $\mathcal{H}$ (i.e.,for any $\alpha, \beta \in \mathbb{C}$ and any $k_1, k_2 \in \mathcal{K}$, we have $\alpha k_1 + \beta k_2 \in \mathcal{K}$). If $\mathcal{K}$ is also closed under the norm of $\mathcal{H}$, (i.e., any $\{k_n\}_{n=1}^{\infty} \subset \mathcal{K}$, if $\{k_n\}_{n=1}^{\infty}$ converges in the norm of $\mathcal{H}$ to some $k \in \mathcal{H}$, then $k \in \mathcal{K}$.) Then $\mathcal{K}$ is called a closed subspace of $\mathcal{H}$.*

We often omit *closed* and call such $\mathcal{K}$ a subspace of $\mathcal{H}$. Now let us turn to the specific Hilbert space $L^2(\mathbb{R})$ and define a sequence of subspaces of $L^2(\mathbb{R})$ which are related to Haar wavelet.(The connection will not be clear until later chapters.)

Example 1 For any $n \in \mathbb{Z}$, define a subset of $L^2(\mathbb{R})$ as follows: Let $V_n = \{f \in L^2(\mathbb{R}) \big| \text{for any } j \in \mathbb{Z}, f|_{[\frac{j}{2^n}, \frac{j+1}{2^n})} \text{ is constant } \}$.

It is trivial to check that each V_n is a linear subspace. To see that V_n is closed under the $L^2(\mathbb{R})$ norm is beyond the scope of our course. We will accept this fact.

Clearly, we see that for any $n \in \mathbb{Z}$, $V_n \subset V_{n+1}$, and $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$. We leave the proof of these fact to the reader. Several other properties are less trivial, we treat them in the following lemma. For any subset A of a Hilbert space $\mathcal{H}$, we use $\overline{A}$ to denote the closure of A in $\mathcal{H}$ under the norm of $\mathcal{H}$, (i.e., $x \in \overline{A} \Longleftrightarrow$ for any $\varepsilon > 0$, there is a $y \in A$ such that $||x - y||_{\mathcal{H}} < \varepsilon$.) Using this terminology, we can rephrase Lemma 3 of last chapter to say that $\overline{V} = L^2(\mathbb{R})$.

Lemma 1. *For any $n \in \mathbb{Z}$, let $V_n = \{f \in L^2(\mathbb{R}) \big| \text{for any } j \in \mathbb{Z}, f|_{[\frac{j}{2^n}, \frac{j+1}{2^n})} \text{ is constant } \}$. Then*

(a)for any $n \in \mathbb{Z}$, $V_n \subset V_{n+1}$.

(b)for any $n \in \mathbb{Z}$, $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$.

(c)$\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$,

(d)$\cap_{n\in\mathbb{Z}} V_n = \{0\}$.

(e)if

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & otherwise \end{cases}$$

then $\varphi(x) \in V_0$ *and* $\{\varphi(x-l) | l \in \mathbb{Z}\}$ *is a complete orthonormal system of* V_0.

Proof. (c)Clearly $\overline{\cup_{n\in\mathbb{Z}} V_n} \subset L^2(\mathbb{R})$, so we only need to show $L^2(\mathbb{R}) \subset \overline{\cup_{n\in\mathbb{Z}} V_n}$. By Lemma 3 of Last chapter, for any $f \in L^2(\mathbb{R})$, any $\varepsilon > 0$, there is $g \in V$, such that $||f-g|| < \varepsilon$, where V is defined as in last chapter. For such $g \in V$, according to the definition of V, there is $m \in \mathbb{Z}$, such that $f|_{[\frac{j}{2^m}, \frac{j+1}{2^m})}$ is constant for any $j \in \mathbb{Z}$. This means $f \in V_m \subset \cup_{n\in\mathbb{Z}} V_n$. In short, for any $f \in L^2(\mathbb{R})$, any $\varepsilon > 0$, there is $g \in \cup_{n\in\mathbb{Z}} V_n$, such that $||f-g|| < \varepsilon$, so $f \in \overline{\cup_{n\in\mathbb{Z}} V_n}$.

(d)Clearly $\{0\} \subset \cap_{n\in\mathbb{Z}} V_n$, so we only need to show $\cap_{n\in\mathbb{Z}} V_n \subset \{0\}$. Recall that for each pair of $n, l \in \mathbb{Z}$,

$$H_{n,l}(x) = \begin{cases} 2^{\frac{n}{2}} & x \in [\frac{2l}{2^{n+1}}, \frac{2l+1}{2^{n+1}}) \\ -2^{\frac{n}{2}} & x \in [\frac{2l+1}{2^{n+1}}, \frac{2l+2}{2^{n+1}}) \\ 0 & otherwise. \end{cases}$$

Observe that for any fixed $n \in \mathbb{Z}$, the inner product of $H_{n,l}$ with any function in V_n is 0 for any $l \in \mathbb{Z}$. Now suppose $f \in \cap_{n\in\mathbb{Z}} V_n$, then $f \in V_n$ for each $n \in \mathbb{Z}$. Hence For each $n \in \mathbb{Z}$, $\langle f, H_{n,l} \rangle = 0$ for all $l \in \mathbb{Z}$. Since we have already proved that $\{H_{n,l} | n, l \in \mathbb{Z}\}$ is a complete orthonormal system in $L^2(\mathbb{R})$, this means that $f = 0$. Hence $\cap_{n\in\mathbb{Z}} V_n \subset \{0\}$.

The proof of (e) is left to the reader. $\square$

Generalizing above sequence of subspaces related to Haar wavelet, we have the concept of **Multiresolution Analysis**.

Definition 2. *For any* $n \in \mathbb{Z}$, *let* V_n *be a subspace of* $L^2(\mathbb{R})$. *Suppose* $\{V_n\}_{n\in\mathbb{Z}}$ *satisfies the following conditions:*

(a)For any $n \in \mathbb{Z}$, $V_n \subset V_{n+1}$;

(b)For any $n \in \mathbb{Z}$, $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$;

(c)$\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$;

(d)$\cap_{n\in\mathbb{Z}} V_n = \{0\}$;

(e)There exists $\varphi(x) \in V_0$ *such that* $\{\varphi(x-l) | l \in \mathbb{Z}\}$ *is a complete orthonormal system of* V_0.

Then $\{V_n\}_{n\in\mathbb{Z}}$ *is called a* **Multiresolution Analysis***.*

Multiresolution Analysis plays an important role in wavelet theory. In order to understand the mechanism involved, we need to know something about orthogonal projection and direct sum decomposition. Next we turn to general Hilbert space to introduce the concept of orthogonal projection. But first we state a simple lemma.

Lemma 2. *Lei* $\mathcal{H}$ *be a Hilbert space,* $f, g \in \mathcal{H}$*. Then*

$$||f+g||^2 + ||f-g||^2 = 2||f||^2 + ||g||^2.$$

The proof is simple and left to the reader.

Proposition 1. *Let* $\mathcal{K}$ *be a subspace of Hilbert space* $\mathcal{H}$*,* $h \in \mathcal{H}$*. Then there is a unique* $k_0 \in \mathcal{K}$*, such that*

$$||h-k_0|| = \inf\{||h-k|| \big| k \in \mathcal{K}\}.$$

Proof. Since $||h-k|| \geq 0$ for any $k \in \mathcal{K}$, so $d = \inf\{||h-k|| \big| k \in \mathcal{K}\}$ exists. By definition of inf we see that

1)$||h-k|| \geq d$ for any $k \in \mathcal{K}$,

2)for any $\varepsilon > 0$, there is a $k_\varepsilon \in \mathcal{K}$ such that $d \leq ||h-k_\varepsilon|| < d+\varepsilon$.

Thus it is not hard to find a sequence $\{k_n\}_{n=1}^\infty \subset \mathcal{K}$ such that $lim_{n\to\infty}||h-k_n|| = d$. Hence $lim_{n\to\infty}||h-k_n||^2 = d^2$. So for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $n > N$,

$$||h-k_n||^2 < d^2 + \frac{\varepsilon^2}{2}.$$

When $n, m > N$, according to Lemma 2,

$$||k_m - k_n||^2 = 2||h-k_n||^2 + 2||h-k_m||^2 - 4||h - \frac{k_n+k_m}{2}||^2$$

$$< d^2 + \frac{\varepsilon^2}{2} + d^2 + \frac{\varepsilon^2}{2} - 4d^2 = \varepsilon^2.$$

Hence $\{k_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$, so there is $k_0 \in \mathcal{H}$, such that $lim_{n\to\infty}||k_n - k_0|| = 0$. Since $\{k_n\}_{n=1}^\infty \in \mathcal{K}$ and $\mathcal{K}$ is a subspace, so $k_0 \in \mathcal{K}$. Now

$$d \leq ||h-k_0|| = ||h-k_n+k_n-k_0|| \leq ||h-k_n|| + ||k_n-k_0||,$$

By taking limit to both side of inequality, we see that $||h-k_0|| = d$.

To see the uniqueness of such k_0, we assume that there is $k_o' \in \mathcal{K}$, such that $||h-k_0'|| = ||h-k_0|| = d$. Then since

$$2d \leq ||2(h - \frac{k_0+k_0'}{2})|| = ||h-k_0+h-k_0'|| \leq ||h-k_0|| + ||h-k_0'|| = 2d,$$

so by Lemma 2, $||k_0-k_0'||^2 = 2||h-k_0||^2 + 2||h-k_0'||^2 - ||h-k_0+h-k_0'||^2 = 0$. Therefore $k_0 - k_0'$. $\square$

Recall from Chapter 1, when E is a subset of $\mathcal{H}$, we say that $f \in \mathcal{H}$ is **orthogonal** to E if f is orthogonal to every element in E. We will use $f \perp E$ to denote this situation.

Proposition 2. *Let $\mathcal{K}$ be a subspace of Hilbert space $\mathcal{H}$, $h \in \mathcal{H}$, $k_0 \in \mathcal{K}$. Then the following are equivalent:*

(a)$||h-k_0|| = \inf\{||h-k|| \big| k \in \mathcal{K}\}$.

(b)$h - k_0 \perp \mathcal{K}$.

Proof. (a)$\Longrightarrow$(b) Let $d = \inf\{||h-k|| \big| k \in \mathcal{K}\}$, then $||h-k_0|| = d$. For any $0 \neq k \in \mathcal{K}$, since $\mathcal{K}$ is a subspace, certainly $k_0 - k \in \mathcal{K}$. So

$$d^2 \leq ||h-k_0+k||^2 = \langle h-k_0+k, h-k_0+k\rangle$$

$$= ||h-k_0||^2 + 2Re\langle h-k_0, k\rangle + ||k||^2 = d^2 + 2Re\langle h-k_0,k\rangle + ||k||^2.$$

So

$$\frac{-2Re\langle h-k_0,k\rangle}{||k||^2} \leq 1.$$

In the argument above, if we replace k with εk for any $\varepsilon > 0$, the computation still holds, and we get

$$\frac{-2Re\langle h-k_0,\varepsilon k\rangle}{||\varepsilon k||^2} \leq 1$$

or

$$\frac{-2Re\langle h-k_0,k\rangle}{||k||^2} \leq \varepsilon.$$

Hence $Re\langle h-k_0,k\rangle = 0$. Likewise, we can replace k with $i\varepsilon k$, similarly, we get $Im\langle h-k_0,k\rangle = 0$. So $\langle h-k_0,k\rangle = 0$ for all $k \in \mathcal{K}$.

(b)$\Longrightarrow$(a)For any $k \in \mathcal{K}$, since $\mathcal{K}$ is a subspace, certainly $k_0 - k \in \mathcal{K}$. Hence $\langle h-k_0, k_0-k\rangle = 0$. Thus

$$||h-k||^2 = ||h-k_0+k_0-k||^2 = ||h-k_0||^2 + ||k_0-k||^2 \geq ||h-k_0||^2.$$

Thus it is easily checked that $||h-k_0|| = \inf\{||h-k|| \big| k \in \mathcal{K}\}$. $\square$

Remark When $\mathcal{K}$ is a subspace of Hilbert space $\mathcal{H}$, for any $h \in \mathcal{H}$, there is a unique $k_0 \in \mathcal{K}$ such that $h - k_0 \perp \mathcal{K}$ and $h = k_0 + h - k_0$. If we consider the subset of $\mathcal{H}$ defined by $\mathcal{M} = \{x \in \mathcal{H} \mid x \perp \mathcal{K}\}$, then it can be checked that $\mathcal{M}$ is a linear subspace of $\mathcal{H}$. Furthermore, if a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{M}$ and $\lim_{n\to\infty} ||x_n - x|| = 0$ for some $x \in \mathcal{H}$, then $x \in \mathcal{M}$. In other words, $\mathcal{M}$ is closed under the norm of $\mathcal{H}$. Hence $\mathcal{M}$ is a (closed) subspace of $\mathcal{H}$. The above two proposition says that for any $h \in \mathcal{H}$, there is a unique $k_0 \in \mathcal{K}$, a unique $m_0 = h - k_0 \in \mathcal{M}$ and $h = k_0 + m_0$.

Definition 3. *Suppose that $\mathcal{K}$ is a subspace of Hilbert space $\mathcal{H}$, $h \in \mathcal{H}$. Then the unique element $k_0 \in \mathcal{K}$ satisfying $h - k_0 \perp \mathcal{K}$ (equivalently $||h - k_0|| = \inf\{||h - k|| | k \in \mathcal{K}\}$.) is call the* **orthogonal projection** *of h onto subspace $\mathcal{K}$. We denote $P_{\mathcal{K}}h = k_0$. The (closed) subspace $\mathcal{M} = \{x \in \mathcal{H} \mid x \perp \mathcal{K}\}$ is called the* **orthogonal complement** *of $\mathcal{K}$ in $\mathcal{H}$. We denote $\mathcal{M} = \mathcal{H} \ominus \mathcal{K}$.*

Remark When $\mathcal{K}$ is a subspace of Hilbert space $\mathcal{H}$, $\mathcal{M} = \mathcal{H} \ominus \mathcal{K}$, let us investigate the subspace $\mathcal{H} \ominus \mathcal{M}$. For any $h \in \mathcal{H}$, by definition of orthogonal complement, $h \in \mathcal{H} \ominus \mathcal{M}$ if and only if $h \perp \mathcal{M}$. By definition of $\mathcal{M}$, certainly for any $k \in \mathcal{K}$, we have $k \perp \mathcal{M}$. This means $\mathcal{K} \subset \mathcal{H} \ominus M$. For $\mathcal{H} \ominus M \subset \mathcal{K}$, consider any $h \in \mathcal{H}$ with $h \perp \mathcal{M}$, since there is a unique $h_{\mathcal{K}} \in \mathcal{K}$, a unique $h_{\mathcal{M}} \in \mathcal{M}$ such that $h = h_{\mathcal{K}} + h_{\mathcal{M}}$, we see that $0 = \langle h, h_{\mathcal{M}} \rangle = \langle h_{\mathcal{K}}, h_{\mathcal{M}} \rangle + \langle h_{\mathcal{M}}, h_{\mathcal{M}} \rangle = ||h_{\mathcal{M}}||^2$. So $h_{\mathcal{M}} = 0$ and $h = h_{\mathcal{K}} \in \mathcal{K}$. Thus $\mathcal{K} = \mathcal{H} \ominus M = \mathcal{H} \ominus (\mathcal{H} \ominus \mathcal{K})$. Thus orthogonal complement is a symmetric relation.

Definition 4. *Suppose that $\mathcal{K}$, $\mathcal{M}$ are subspace of Hilbert space $\mathcal{H}$. If $\mathcal{K}$, $\mathcal{M}$ are orthogonal complement of each other in $\mathcal{H}$, then we say $\mathcal{H}$ is the direct sum of $\mathcal{K}$ and $\mathcal{M}$. We denote $\mathcal{H} = \mathcal{K} \oplus \mathcal{M}$.*

Lastly, we consider the problem of how to calculate the orthogonal projection of h onto any subspace $\mathcal{K}$ of Hilbert space $\mathcal{H}$. We will see, once we have a complete orthonormal system of $\mathcal{K}$, this can be easily done.

Proposition 3. *Let $\mathcal{H}$ be a Hilbert space with subspace $\mathcal{K}$, $\{e_j\}_{j\in\mathbb{J}}$ is a complete orthonormal system of $\mathcal{K}$, where $\mathbb{J}$ is any countable index set. Then for $\forall h \in \mathcal{H}$,*

$$P_{\mathcal{K}}h = \sum_{j\in\mathbb{J}} \langle h, e_j \rangle e_j.$$

Proof. First of all, according to Bessel's inequality in Chapter 1,

$$\sum_{j\in\mathbb{J}} |\langle h, e_n \rangle|^2 < \infty.$$

Hence by Corollary 1 in Chapter 1, $\sum_{j\in\mathbb{J}}\langle h, e_j\rangle e_j$ converges in $\mathcal{H}$. Furthermore, since $\mathcal{K}$ is subspace of $\mathcal{H}$, $\{e_j\}_{j\in\mathbb{J}} \subset \mathcal{K}$, so actually $\sum_{j\in\mathbb{J}}\langle h, e_j\rangle e_j$ converges in some element in $\mathcal{H}$. Let us denote

$$g = \sum_{j\in\mathbb{J}}\langle h, e_j\rangle e_j.$$

Now we do some computations using Lemma 3 of Chapter 1. For any $j \in \mathbb{J}$, we see

$$\langle g, e_l\rangle = \sum_{j\in\mathbb{J}}\langle h, e_j\rangle\langle e_j, e_l\rangle = \langle h, e_l\rangle.$$

This means that for any $j \in \mathbb{J}$,

$$\langle h - g, e_j\rangle = \langle h, e_j\rangle - \langle g, e_j\rangle = 0.$$

Since $h - g$ is orthogonal to each element of a complete orthonormal system of $\mathcal{K}$, we see that $h - g \perp \mathcal{K}$. According to Proposition 2, g is the unique element such that $||h - g|| = \inf\{||h - k|| \big| k \in \mathcal{K}\}$. That is to say, $g = P_{\mathcal{K}}h$. □

Sometimes we need to compute orthogonal projection of some elements onto different subspaces. The following is a case often encountered.

Proposition 4. *Let $\mathcal{H}$ be a Hilbert space with subspace $\mathcal{K}_1$, $\mathcal{K}_2$. If $\mathcal{K}_1 \subset \mathcal{K}_2$, then for $\forall h \in \mathcal{H}$,*

$$P_{\mathcal{K}_1}h = P_{\mathcal{K}_1}(P_{\mathcal{K}_2}h).$$

Proof. Let $\{e_j\}_{j\in\mathbb{J}}$ is a complete orthonormal system of $\mathcal{K}_1$, where $\mathbb{J}$ is any countable index set. According to Proposition 3, we have for $\forall h \in \mathcal{H}$,

$$P_{\mathcal{K}_1}h = \sum_{j\in\mathbb{J}}\langle h, e_j\rangle e_j,$$

$$P_{\mathcal{K}_1}(P_{\mathcal{K}_2}h) = \sum_{j\in\mathbb{J}}\langle P_{\mathcal{K}_2}h, e_j\rangle e_j.$$

Note that $h - P_{\mathcal{K}_2}h \perp \mathcal{K}_2$. Since $\mathcal{K}_1 \subset \mathcal{K}_2$, we have $h - P_{\mathcal{K}_2}h \perp \mathcal{K}_1$. Since $\{e_j\}_{j\in\mathbb{J}}$ is a complete orthonormal system of $\mathcal{K}_1$, so $\forall j \in \mathbb{J}$, we have $\langle h - P_{\mathcal{K}_2}h, e_j\rangle = 0$. So $\langle h, e_j\rangle = \langle P_{\mathcal{K}_2}h, e_j\rangle$ for any $j \in \mathbb{J}$. Hence $P_{\mathcal{K}_1}h = P_{\mathcal{K}_1}(P_{\mathcal{K}_2}h)$. □

This brings Chapter 5 to the end. In next chapter we will apply what we have learned here to a specific structure in $L^2(\mathbb{R})$, namely the so called Multiresolution Analysis.